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DEVELOPMENT OF THE EQUATIONS OF MOTION AND TRANSFER FUNCTIONS F--ETC(U)
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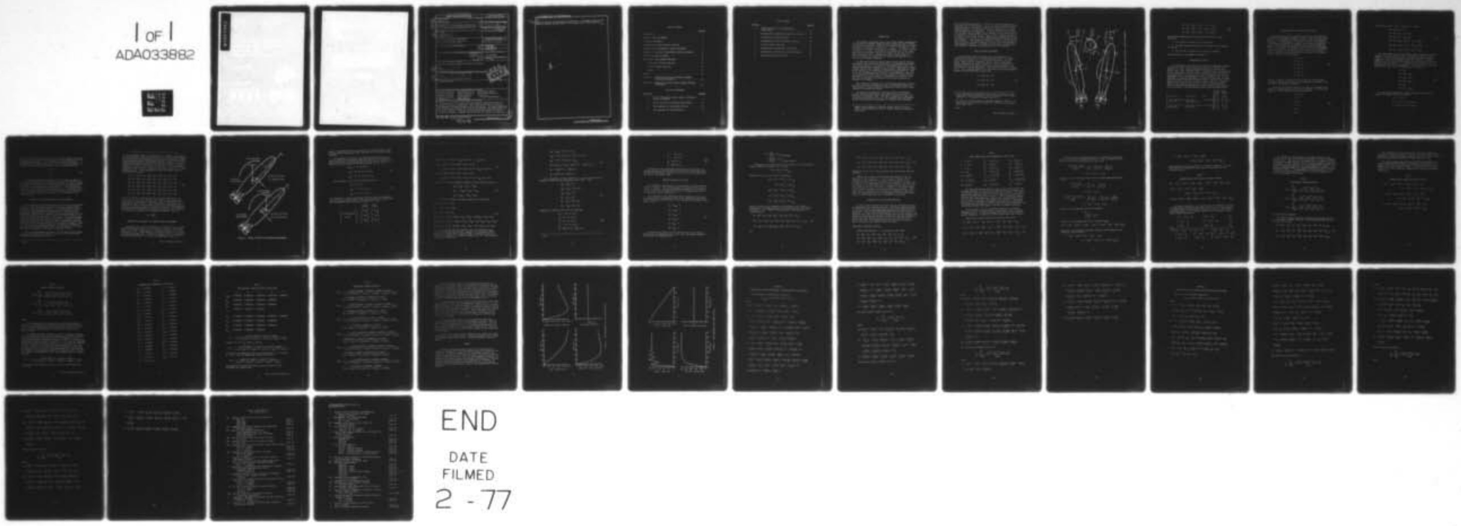

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

> Derivation of the linear, small-perturbation equations of motion for underwater vehicles is presented. These equations include the inertial, hydrodynamic, and gravity-buoyancy forces and moments. Necessary assumptions are introduced to linearize the equations and decouple the longitudinal from the lateral motions. Solutions to the equations are obtained by Laplace transform techniques. Complete expressions for the transfer functions are

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given in terms of the hydrodynamic coefficients. An example problem is presented to provide typical values for the characteristic vehicle motions.

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INTRODUCTION

The economic or military value of any vehicle depends fundamentally on its ability to traverse a specific path between its point of departure and its destination. Means for the control of the path vary widely and depend on a variety of constraints. A train, for example, is constrained to move along a track but is not steered. Its control freedom is merely one of speed. An automobile or surface ship, on the other hand, while constrained to move on the surface of the land or the sea, must be steered as well. Underwater vehicles, however, have an additional degree of freedom; for this reason their problems of the control are of unusual complexity.

An underwater vehicle or weapon system contains spatial sensors, and guidance and control devices (possibly all contained in the human pilot) whose purpose it is to develop three-dimensional flight path commands appropriate to steering so as to reach a destination or target, and then execute those commands by maintaining or modifying the forces on the vehicle so as to maintain or modify the velocity vector. This allows the intended purpose or mission to be accomplished. Qualities of a vehicle that tend to make it resist changes in the direction or magnitude of its velocity vector are referred to as stability, while the ease and expedition with which the vector may be altered are referred to as control. [1]

The purpose of this report is to develop the equations of motions and their associated transfer functions for underwater vehicles in a form suitable for use by engineers charged with solving these complex problems of stability and control.

One begins by equating the forces and moments (hydrodynamic, gravity, and buoyancy) acting on the vehicle to its reactions, in accordance with Newton's laws. The body is assumed to be rigid with the axis system located at its c.g. The external forces and moments acting on the vehicle are represented by a Taylor series expansion

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[1] McRuer, Duane; Ashkenas, Irving; and Graham, Dunston, *Aircraft Dynamics and Automatic Control*, pp. 203-220, Princeton University Press, 1973.

about an equilibrium condition. The theory of small perturbations is introduced into the derivation to arrive at linear decoupled equations. These equations can be solved in either the time domain [2] or in the frequency domain. The technique of frequency domain solution using Laplace transforms is chosen here. It is then possible to use convenient transfer function models for the dynamics of the vehicle and all the analytical techniques for the study of linear feedback systems can be brought to bear on the problem. Appropriate nondimensional factors are introduced to conform to the standard nomenclature followed in Navy publications. [3] The complete six degree of freedom, small perturbation, linearized, decoupled equations of motion are then presented for a self-propelled vehicle. An example is given using a typical underwater vehicle to demonstrate the application of the equations.

INERTIAL FORCES AND MOMENTS

The equations of motion employed are a linearization of the classical Euler equations for a rigid body with respect to a set of axes fixed at the body c.g. The positive directions of the axes, angles, linear velocity components, angular velocity components, forces, and moments are shown in Figure 1. Since the derivation of these equations can be found in any standard text on dynamics, the details will not be presented here. The reader should refer to McRuer, et al.[1] for a complete derivation of the inertial forces and moments. Referred to body-fixed axes one may write

$$\begin{aligned}
\Sigma X &= m(\dot{U} + QW - RV) \\
\Sigma Y &= m(\dot{V} + RU - PW) \qquad (1) \\
\Sigma Z &= m(\dot{W} + PV - QU)
\end{aligned}$$

---

[2] Naval Ship Research Development Center, Report P-433-M-01, *Users Guide, NSRDC Digital Program for Simulating Submarine Motion ZZMN-Revision 1.0,* by Ronald W. Richards, June 1971.

[3] The Society of Naval Architects and Marine Engineers, TMB No. 1-5, *Nomenclature for Treating the Motions of a Submerged Body Through a Fluid,* April 1952.

[1] ibid.

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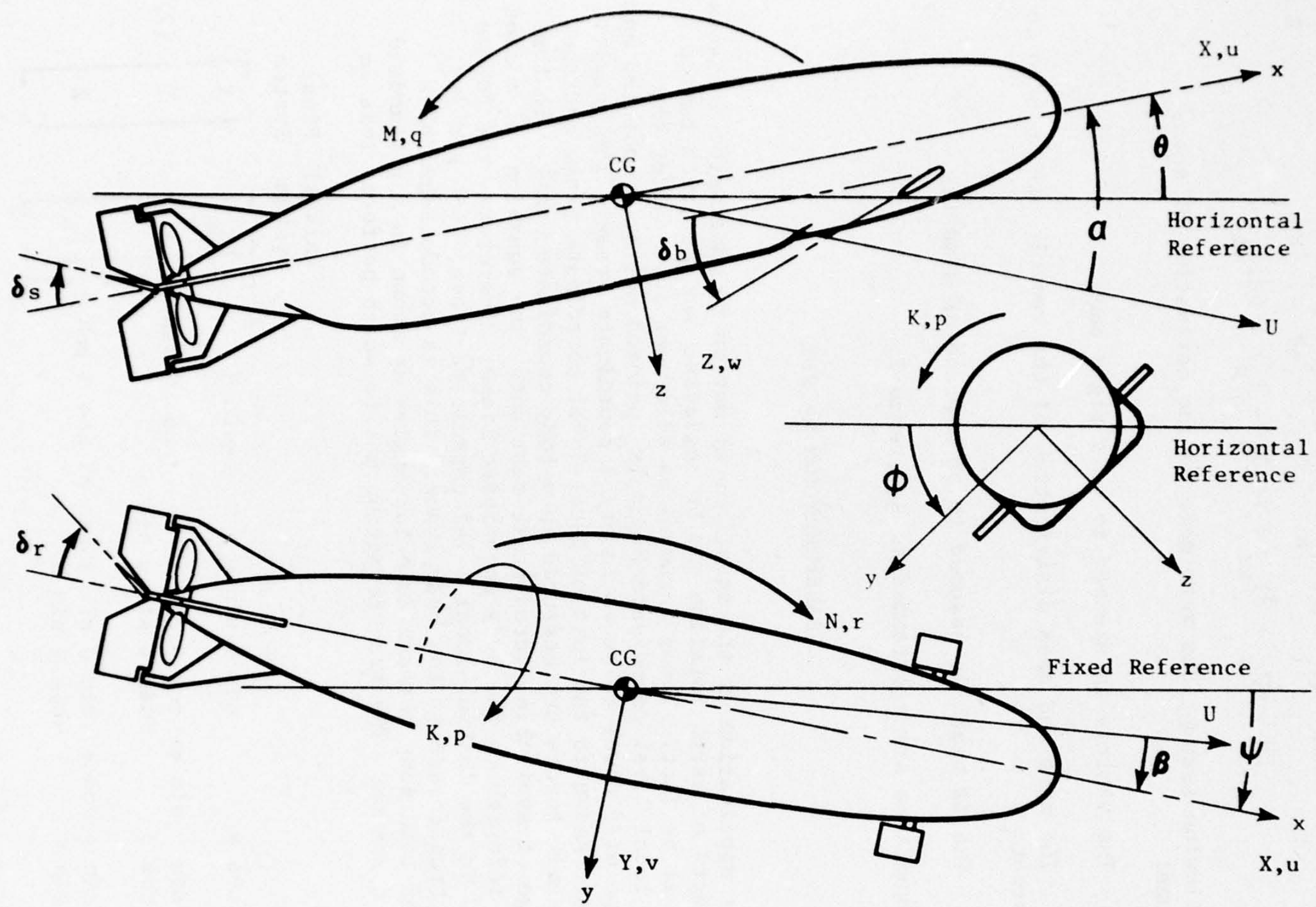

FIGURE 1. POSITIVE DIRECTIONS OF AXES, ANGLES, VELOCITIES, FORCES AND MOMENTS

$$\Sigma K = \dot{P}I_x - \dot{R}I_{xz} + QR(I_z - I_y) - PQI_{xz}$$

$$\Sigma M = \dot{Q}I_y + PR(I_x - I_z) - R^2 I_{xz} + P^2 I_{xz} \qquad \text{(1) Con.}$$

$$\Sigma N = \dot{R}I_z - \dot{P}I_{xz} + PQ(I_y - I_x) + QRI_{xz}.$$

The following assumptions were made in the derivation of these equations:

1. The vehicle is assumed to be a rigid body.

2. The mass and mass distribution of the vehicle are assumed to be constant.

3. The XZ plane is assumed to be a plane of symmetry.

These equations are programmed in Reference 2.

TRANSFORMATION MATRIX

For application of the equations of motion to underwater vehicles, the effects of earth rotation can be neglected and the earth can be assumed to be flat. These assumptions allow one to consider the initial local level coordinate frame (x_o pointed north, y_o pointed east, and z_o pointed down) to be an inertial coordinate frame. The gravity forces are known in the initial local level coordinate frame and the hydrodynamic forces are measured in a body coordinate frame. To take advantage of available hydrodynamic test data, the equations of motion will be referred to the body coordinate frame. Therefore, the forces measured in the initial local level coordinate frame, i.e. gravity, must be transformed to the body frame. This is accomplished by a series of rotations through Eulerian angles as shown in any standard text on dynamics. The transformation matrix which performs this is

$$\begin{array}{cccc} & & \text{Initial System} & \text{Final System} \\ \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix} & & \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} = & \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \end{array} \qquad (2)$$

LINEARIZATION OF THE EQUATIONS OF MOTION

Equations (1) are to be linearized before they are expanded to include the hydrodynamic and gravity forces. These equations contain products of the dependent variables, therefore they are in general nonlinear. To reduce them to a tractable form, the total motion can be considered as composed of two parts: an average or mean motion that is representative of the operating point or equilibriums conditions, and a dynamic motion that accounts for small perturbations about the mean motion. Accordingly, each of the total instantaneous velocity components of the vehicle can be written as the sum of a velocity component during the equilibrium condition and a change in velocity caused by the disturbance:

$$
\begin{aligned}
U &= U_o + u \\
V &= V_o + v \\
W &= W_o + w \\
P &= P_o + p \\
Q &= Q_o + q \\
R &= R_o + r
\end{aligned}
\tag{3}
$$

The zero subscripts in Equations (3) indicate the steady flight velocities, and the lower case letters represent the changes in the velocities (disturbance velocities).

During the equilibrium condition, the vehicle is assumed to be flying with wings level and with all the components of velocity zero except U_o. Thus, Equation (3) may be rewritten as

$$
\begin{aligned}
U &= U_o + u \\
V &= v \\
W &= w \\
P &= p \\
Q &= q \\
R &= r
\end{aligned}
\tag{4}
$$

Substituting Equations (4) in Equations (1) yields

$$
\begin{aligned}
\Sigma X &= m(\dot{u} + qw - rv) \\
\Sigma Y &= m(\dot{v} + rU_o + ru - pw) \\
\Sigma Z &= m(\dot{w} + pv - qU_o - qu) \\
\Sigma K &= \dot{p}I_x - \dot{r}I_{xz} + qr(I_z - I_y) - pqI_{xz} \\
\Sigma M &= \dot{q}I_y + pr(I_x - I_z) - r^2I_{xz} + p^2I_{xz} \\
\Sigma N &= \dot{r}I_z - \dot{p}I_{xz} + pq(I_y - I_x) + qrI_{xz}.
\end{aligned}
\tag{5}
$$

The disturbances from the equilibrium flight condition will now be assumed to be small enough so that the products and squares of the changes in velocities are negligible in comparison with the changes themselves. Also, the disturbance angles are assumed to be small enough so that the sines of these angles may be set equal to the angles and the cosines set equal to 1. Products of these angles are also approximately zero and can be neglected. Thus, terms similar to rv and uq may be set equal to zero, and Equations (5) reduced to

$$
\begin{aligned}
\Sigma X &= m\dot{u} \\
\Sigma Y &= m(\dot{v} + rU_o) \\
\Sigma Z &= m(\dot{w} - qU_o) \\
\Sigma K &= \dot{p}I_x - \dot{r}I_{xz} \\
\Sigma M &= \dot{q}I_y \\
\Sigma N &= \dot{r}I_z - \dot{p}I_{xz}.
\end{aligned}
\tag{6}
$$

The relationship between the angular velocities and the rate of change of the angles $(\dot{\phi},\dot{\theta},\dot{\psi})$ are given by

$$
\begin{aligned}
P &= \dot{\phi} - \dot{\psi}\sin\theta \\
Q &= \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\
R &= \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi
\end{aligned}
$$

where P, Q, and R are the angular velocities measured about the vehicle body axis and $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ are the angular velocities measured about the equilibrium axis system. P, Q, and R are the angular velocities measured by a rate gyro located in the vehicle. Applying Equations (4) and the small perturbation assumption, these equations reduce to

$$
\begin{aligned}
p &= \dot{\phi} \\
q &= \dot{\theta} \qquad\qquad (7) \\
r &= \dot{\psi}.
\end{aligned}
$$

The small perturbations assumption not only limits the applicability of Equations (6) to small changes in angles and velocities about an equilibrium condition, but reduces Equations (6) to linear equations and yields a simplification of the mathematical methods necessary for the analysis of the complicated vehicle motions. In a rigorous mathematical sense, Equations (6) are applicable only to infinitesimal disturbances; however, experience has shown that quick and accurate results can be obtained by applying these equations to disturbances of more finite magnitude.

EXPANSION OF THE HYDRODYNAMIC FORCES AND MOMENTS

The hydrodynamic forces are exerted on the vehicle by the surrounding fluid. They are present whenever there are any reactive forces between the fluid mass and the vehicle. In steady flight they result from relative motion between the vehicle and the fluid mass or from accelerated flows produced by the deflection of a control surface. Although the specific forces depend on their peculiar origins, the form of the expressions that describe perturbations in these forces is not particularly dependent upon origin. For example, on vehicles designed to be propelled at very low speeds by divers, the dominant forces and moments are produced by the accelerated flows (added mass and moments of inertia) and the gravity-buoyancy terms (metacentric moment). Whereas in the case of a high speed submarine, the dominant contributions come from the steady hydrodynamic terms. The end results of the treatment here are pertinent to all kinds of underwater vehicles.

These forces and moments are known to be functions of the relative velocity, acceleration, and position as well as control deflections[1]. In functional form they are written as

---

[1] ibid.

$$F = f(\dot{u}, \dot{v}, \dot{w}, u, v, w, \dot{q}, q, \theta, \dot{p}, p, \phi, \dot{r}, r, \psi, \delta).$$

If the hydrodynamic forces are considered to be continuous functions of all these variables, each of the forces (X_h, Y_h, and Z_h) and the moments (K_h, M_h, and N_h) can be expressed in terms of the variables by a Taylor series expansion about the equilibrium condition. Because of the small perturbation assumptions, second order and higher terms of the Taylor series are neglected. Additionally, because the XZ plane is a plane of symmetry, X_h, Z_h, and M_h are functions of only u, w, q, their derivatives and θ, whereas Y_h, K_h, and N_h are functions of only v, p, r, their derivatives and ϕ. Thus, expanding the forces and moments about an equilibrium point yields

$$\begin{aligned}
Y_h &= Y_o + Y_{\dot{v}}\dot{v} + Y_v v + Y_{\dot{p}}\dot{p} + Y_p p + Y_\phi \phi + Y_r r + Y_{\dot{r}}\dot{r} + Y_\delta \delta \\
K_h &= K_o + K_{\dot{v}}\dot{v} + K_v v + K_{\dot{p}}\dot{p} + K_p p + K_\phi \phi + K_{\dot{r}}\dot{r} + K_r r + K_\delta \delta \\
N_h &= N_o + N_{\dot{v}}\dot{v} + N_v v + N_{\dot{p}}\dot{p} + N_p p + N_\phi \phi + N_{\dot{r}}\dot{r} + N_r r + N_\delta \delta \\
X_h &= X_o + X_{\dot{u}}\dot{u} + X_u u + X_{\dot{w}}\dot{w} + X_w w + X_{\dot{q}}\dot{q} + X_q q + X_\theta \theta + X_\delta \delta \\
Z_h &= Z_o + Z_{\dot{u}}\dot{u} + Z_u u + Z_{\dot{w}}\dot{w} + Z_w w + Z_{\dot{q}}\dot{q} + Z_q q + Z_\theta \theta + Z_\delta \delta \\
M_h &= M_o + M_{\dot{u}}\dot{u} + M_u u + M_{\dot{w}}\dot{w} + M_w w + M_{\dot{q}}\dot{q} + M_q q + M_\theta \theta + M_\delta \delta
\end{aligned} \qquad (8)$$

Each of the terms in Equations (8) has a physical significance. X_o, Y_o, Z_o, K_o, M_o, and N_o are the forces and moments acting along and about the X, Y, and Z axes, respectively, while the vehicle is in the equilibrium flight condition. The terms similar to $X_u u$ express the change in the given force and moment caused by the disturbance quantity. The term X_u is known as either a stability derivative or a hydrodynamic coefficient and is defined as the change in the X-force with respect to the u-velocity, evaluated at the equilibrium condition. That is,

$$X_u = \left(\frac{\partial X}{\partial u}\right)_o .$$

EXPANSION OF THE GRAVITY AND BUOYANCY FORCES AND MOMENTS

The gravity force can be considered to act at the center of gravity, whereas the buoyant force acts at the center of buoyancy. The buoyant force also creates a moment since it is displaced from the c.g. by the distance X_B, Y_B, Z_B. These distances are shown in Figure 2 for a stable and unstable configuration. To find the expressions for the components of gravity and buoyancy to be used in the equations of motion, the

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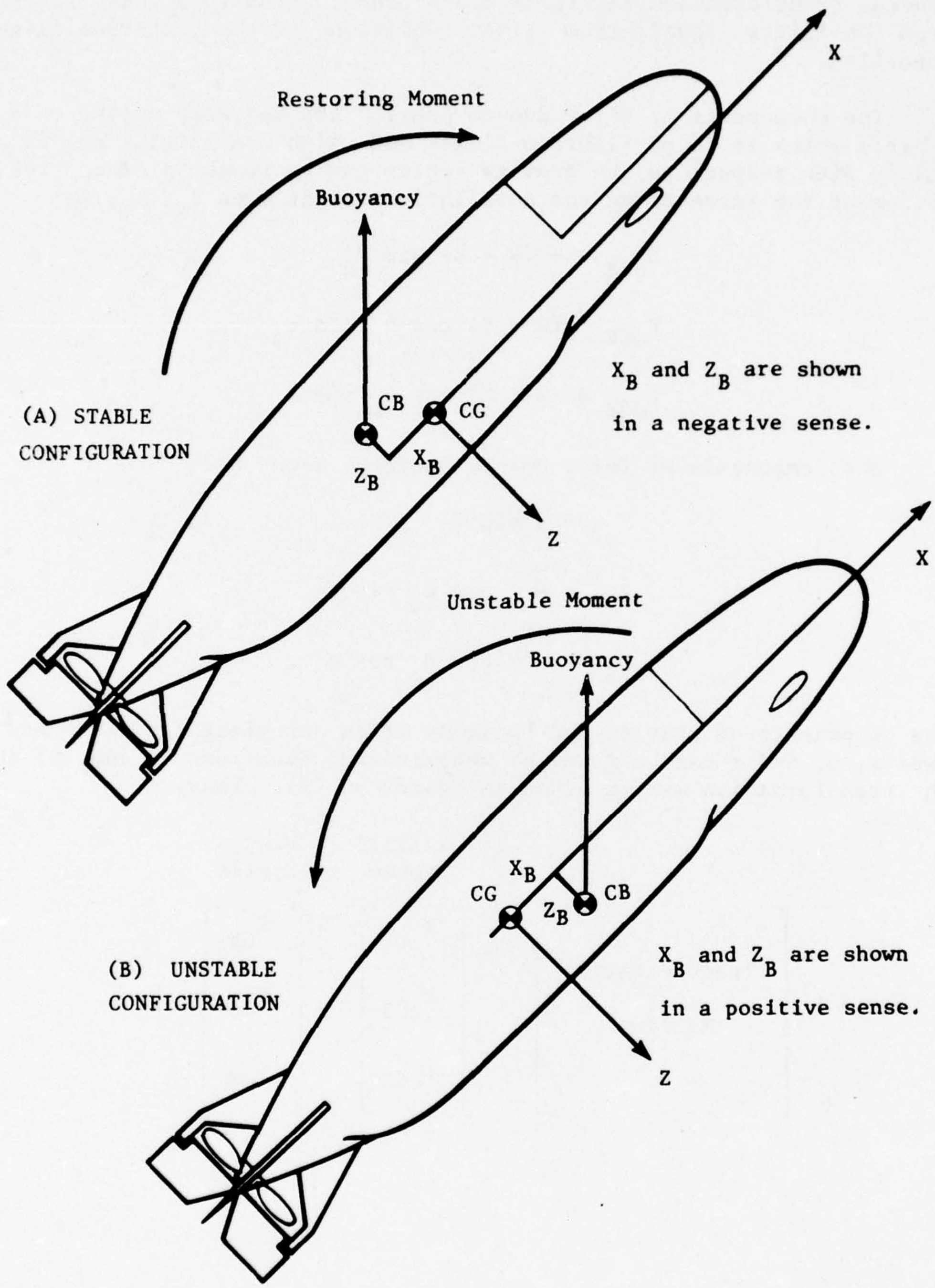

FIGURE 2. CENTER OF GRAVITY AND BUOYANCY RELATIONSHIP

general transformation matrix is again used to transform these forces from the initial equilibrium flight conditions to the disturbed flight condition.

The components of force due to gravity and buoyancy acting on a vehicle which is in equilibrium flight and which has initial angles θ_o and ϕ_o with respect to the gravity vector can be found by direct resolution of the force along the equilibrium flight axes X_o, Y_o, Z_o:

$$\begin{aligned} X_{oGB} &= -(W - B)\sin\theta_o \\ Y_{oGB} &= (W - B)\cos\theta_o \sin\phi_o \\ Z_{oGB} &= (W - B)\cos\theta_o \cos\phi_o . \end{aligned} \tag{9}$$

The components of force due to buoyancy alone are

$$\begin{aligned} X_{oB} &= B\sin\theta_o \\ Y_{oB} &= -B\cos\theta_o \sin\phi_o \\ Z_{oB} &= -B\cos\theta_o \cos\phi_o . \end{aligned} \tag{10}$$

The components of gravity and buoyancy which act along the disturbed axes x, y, and z can be found by substituting Equations (9 and 10) into the transformation matrix given in Equations (2). Thus,

$$\begin{array}{ccc} & \text{Initial System} & \text{Final System} \\ \begin{bmatrix} \\ \text{Transformation} \\ \text{Matrix} \\ \\ \end{bmatrix} & \begin{bmatrix} X_{oGB} \\ Y_{oGB} \\ Z_{oGB} \end{bmatrix} = & \begin{bmatrix} X_{GB} \\ Y_{GB} \\ Z_{GB} \end{bmatrix} \end{array}$$

or

$$X_{GB} = X_{oGB} \cos\psi \cos\theta + Y_{oGB} \sin\psi \cos\theta - Z_{oGB} \sin\theta$$

$$\begin{aligned} Y_{GB} &= X_{oGB}(\cos\psi \sin\theta \sin\phi - \sin\psi \cos\phi) \\ &\quad + Y_{oGB}(\cos\psi \cos\phi + \sin\psi \sin\theta \sin\phi) + Z_{oGB} \cos\theta \sin\phi \\ Z_{GB} &= X_{oGB}(\cos\psi \sin\theta \cos\phi + \sin\psi \sin\phi) \\ &\quad + Y_{oGB}(\sin\theta \cos\phi \sin\psi - \sin\phi \cos\psi) + Z_{oGB} \cos\theta \cos\phi \end{aligned} \tag{11}$$

Applying the small perturbation assumption, these equations reduce to

$$\begin{aligned} X_{GB} &= X_{oGB} + \psi Y_{oGB} - \theta Z_{oGB} \\ Y_{GB} &= -\psi X_{oGB} + Y_{oGB} + \phi Z_{oGB} \\ Z_{GB} &= \theta X_{oGB} - \phi Y_{oGB} + Z_{oGB} \,. \end{aligned} \tag{12}$$

In a similar fashion, the moment equations are found from

$$\begin{aligned} X_B &= X_{oB} + \psi Y_{oB} - \theta Z_{oB} \\ Y_B &= -\psi X_{oB} + Y_{oB} + \phi Z_{oB} \\ Z_B &= \theta X_{oB} - \phi Y_{oB} + Z_{oB} \\ M_B &= X_B z_B - Z_B x_B = z_B(X_{oB} + \psi Y_{oB} - \theta Z_{oB}) - x_B(\theta X_{oB} - \phi Y_{oB} + Z_{oB}) \\ N_B &= -X_B y_B + Y_B x_B = -y_B(X_{oB} + \psi Y_{oB} - \theta Z_{oB}) + x_B(-\psi X_{oB} + Y_{oB} + \phi Z_{oB}) \\ K_B &= Z_B y_B - Y_B z_B = y_B(\theta X_{oB} - \phi Y_{oB} + Z_{oB}) - z_B(-\psi X_{oB} + Y_{oB} + \phi Z_{oB}) \,. \end{aligned} \tag{13}$$

Since it is assumed that the wings are level in equilibrium flight, $\phi_o = 0$ and thus Y_{oB} will be zero. However, θ_o is arbitrary since a vehicle can climb or dive in steady flight. Furthermore, since the XZ plane is assumed to be a plane of symmetry, $Y_B = 0$. Thus, Equations (12) and (13) reduce to

$$
\begin{aligned}
X_{GB} &= X_{oGB} - \theta(W - B)\cos\theta_o \\
Y_{GB} &= +\psi(W - B)\sin\theta_o + \phi(W - B)\cos\theta_o \\
Z_{GB} &= -\theta(W - B)\sin\theta_o + Z_{oGB} \\
M_B &= X_{oB}\, z_B - x_B\, Z_{oB} + \theta z_B B\cos\theta_o - \theta x_B B\sin\theta_o \\
N_B &= -x_B B\psi\sin\theta_o - x_B B\phi\cos\theta_o \\
K_B &= z_B\psi B\sin\theta_o + z_B\phi B\cos\theta_o\ .
\end{aligned}
\tag{14}
$$

It is conventional in Navy literature [3] to treat the gravity and buoyancy contributions as hydrodynamic coefficients. Thus,

$$
\begin{aligned}
X_{GB} &= X_{oGB} + X_\theta\theta \\
Y_{GB} &= Y_{oGB} + Y_\phi\phi + Y_\psi\psi \\
Z_{GB} &= Z_{oGB} + Z_\theta\theta \\
M_B &= M_{oB} + M_\theta\theta \\
N_B &= N_{oB} + N_\phi\phi + N_\psi\psi \\
K_B &= K_{oB} + K_\phi\phi + K_\psi\psi
\end{aligned}
\tag{15}
$$

Comparison of Equations (13) and (14) shows that

$$
\begin{aligned}
X_\theta &= -(W - B)\cos\theta_o \\
Y_\phi &= (W - B)\cos\theta_o \\
Y_\psi &= (W - B)\sin\theta_o \\
Z_\theta &= -(W - B)\sin\theta_o \\
M_\theta &= Bz_B\cos\theta_o - Bx_B\sin\theta_o
\end{aligned}
\tag{16}
$$

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[3] ibid.

$$N_\phi = - x_B B \cos \theta_o$$

$$N_\psi = - x_B B \sin \theta_o$$

$$K_\phi = z_B B \cos \theta_o \qquad (16) \text{ Con.}$$

$$K_\psi = z_B B \sin \theta_o \; .$$

Therefore, in the following section when the contributions are grouped together to form the complete equations of motion, the gravity and buoyancy disturbance terms $X_\theta \theta$, $Y_\phi \phi$, etc, will be included in the hydrodynamic disturbance terms.

COMPLETE EQUATIONS OF MOTION

The individual contributions to the equations of motion have been examined in detail. The complete equations of motion of the vehicle can now be written. This is accomplished by summing the contributions of the external forces (Equations 8 and 15) and equating this sum to the right-hand side of Equation (6).

The equations for the equilibrium flight condition can be found by substituting the equilibrium values of the hydrodynamic and gravity forces and moments into Equations (6) and setting the disturbance terms equal to zero:

$$X_o + X_{oGB} = 0$$

$$Y_o + Y_{oGB} = 0$$

$$Z_o + Z_{oGB} = 0$$

$$K_o + K_{oB} = 0 \qquad (17)$$

$$M_o + M_{oB} = 0$$

$$N_o + N_{oB} = 0$$

The equations of motion for the disturbed vehicle are found by substituting the disturbed values of the forces and moments Equations (8) and (15), into Equations (6):

$$m\dot{u} = \boxed{X_o} + (X_i i)_{\text{Hydrodynamic}}$$

$$+ \boxed{X_{oGB}} + (X_i i)_{\text{Gravity-Buoyancy}}$$

The quantities in the boxes sum to zero because of the equilibrium flight condition of Equations (17). Hence,

$$m\dot{u} = (X_i i)_H + (X_i i)_{GB} .$$

Similarly for the Y, Z, K, M, and N equations

$$m(\dot{w} - U_o\dot{\theta}) = (Z_i i)_H + (Z_i i)_{GB}$$

$$I_y\ddot{\theta} = (M_i i)_H + (M_i i)_{GB}$$

$$m(\dot{v} + U_o\dot{\psi}) = (Y_i i)_H + (Y_i i)_{GB}$$

$$I_x\ddot{\phi} - I_{xz}\ddot{\Psi} = (K_i i)_H + (K_i i)_{GB}$$

$$I_z\ddot{\Psi} - I_{xz}\ddot{\phi} = (N_i i)_H + (N_i i)_{GB}$$

where the terms $(X_i i)_H$ represent the hydrodynamic coefficients of equations (8) and $(X_i i)_{GB}$ represent the gravity-buoyancy coefficients of equations (15). With the force and moment terms expanded the equations become

$$m\dot{u} = X_{\dot{u}}\dot{u} + X_u u + X_{\dot{w}}\dot{w} + X_w w + X_{\dot{q}}\dot{q} + X_q q + X_\theta\theta + X_{\delta_s}\delta_s$$

$$m(\dot{w} - U_o\dot{\theta}) = Z_{\dot{u}}\dot{u} + Z_u u + Z_{\dot{w}}\dot{w} + Z_w w + Z_{\dot{q}}\dot{q} + Z_q q + Z_\theta\theta + Z_{\delta_s}\delta_s \qquad (18)$$

$$I_y\ddot{\theta} = M_{\dot{u}}\dot{u} + M_u u + M_{\dot{w}}\dot{w} + M_w w + M_{\dot{q}}\dot{q} + M_q q + M_\theta\theta + M_{\delta_s}\delta_s$$

and

$$m(\dot{v} + U_o\dot{\psi}) = Y_{\dot{v}}\dot{v} + Y_v v + Y_{\dot{p}}\dot{p} + Y_p p + Y_\phi \phi + Y_{\dot{r}}\dot{r} + Y_r r + Y_{\delta_R}\delta_R$$

$$I_x\ddot{\phi} - I_{xz}\ddot{\Psi} = K_{\dot{v}}\dot{v} + K_v v + K_{\dot{p}}\dot{p} + K_p p + K_\phi \phi + K_{\dot{r}}\dot{r} + K_r r + K_{\delta_R}\delta_R \quad (19)$$

$$I_z\ddot{\Psi} - I_{xz}\ddot{\phi} = N_{\dot{v}}\dot{v} + N_v v + N_{\dot{p}}\dot{p} + N_p p + N_\phi \phi + N_{\dot{r}}\dot{r} + N_r r + N_{\delta_R}\delta_R$$

Equations (18 and 19) are the linearized small perturbation equations of motion.

Examination of these equations shows that Equations (18) are functions of the variables u, θ, and w, whereas Equations (19) are functions of the variables v, r, and p, thus as a result of the assumptions made in the previous analysis, the equations of motion can be treated as two independent sets of three equations. Equations (18) are referred to as the longitudinal or symmetrical equations because, when these motions occur, the plane of symmetry of the vehicle remains in the plane it occupied in the equilibrium flight conditions. Equations (19) are referred to as the lateral or asymmetrical equations. Since the longitudinal motions are independent of the lateral motions, they are treated separately in the remainder of this report.

DERIVATION OF THE TRANSFER FUNCTIONS

In this section, Equations (18) and (19) are converted by the use of determinants into transfer functions. The Laplace transform method of solution is used. The transformed equations are nondimensionalized to conform to the standard SNAME nondimensionalizing factors given in Reference 3. Table 1 shows a representative list of these factors. It should be noted that the resulting transfer functions are nondimensional. Before evaluating the characteristic frequency or the time response of the vehicle, the transfer function must be first converted to the dimensional form. The longitudinal transfer functions are derived first, followed by the lateral transfer functions.

Longitudinal Transfer Functions

Substituting Equations (7) into Equations (18) yields

$$m\dot{u} - X_{\dot{u}}\dot{u} - X_u u - X_{\dot{w}}\dot{w} - X_w w - X_{\dot{q}}\ddot{\theta} - X_q\dot{\theta} - X_\theta\theta = X_{\delta_s}\delta_s$$

$$m\dot{w} - mU_o\dot{\theta} - Z_{\dot{u}}\dot{u} - Z_u u - Z_{\dot{w}}\dot{w} - Z_w w - Z_{\dot{q}}\ddot{\theta} - Z_q\dot{\theta} - Z_\theta\theta = Z_{\delta_s}\delta_s \quad (20)$$

$$I_y\ddot{\theta} - M_{\dot{u}}\dot{u} - M_u u - M_{\dot{w}}\dot{w} - M_w w - M_{\dot{q}}\ddot{\theta} - M_q\dot{\theta} - M_\theta\theta = M_{\delta_s}\delta_s$$

TABLE 1

SNAME NOMENCLATURE FOR NONDIMENSIONAL COEFFICIENTS

$m' = m/\frac{1}{2}\rho\ell^3$	$X'_w = X_w/\frac{1}{2}\rho\ell^2 U_o$	$M'_{\dot{w}} = M_{\dot{w}}/\frac{1}{2}\rho\ell^4$
$s' = s\,\ell/U_o$	$X'_{\dot{q}} = X_{\dot{q}}/\frac{1}{2}\rho\ell^4$	$M'_w = M_w/\frac{1}{2}\rho\ell^3 U_o$
$u' = u/U_o$	$X'_q = X_q/\frac{1}{2}\rho\ell^3 U_o$	$M'_{\dot{q}} = M_{\dot{q}}/\frac{1}{2}\rho\ell^5$
$w' = w/U_o$	$X'_\theta = X_\theta/\frac{1}{2}\rho\ell^2 U_o^2$	$M'_q = M_q/\frac{1}{2}\rho\ell^4 U_o$
$X'_{\dot{u}} = X_{\dot{u}}/\frac{1}{2}\rho\ell^3$	$X'_{\delta_s} = X_{\delta_s}/\frac{1}{2}\rho\ell^2 U_o^2$	$M'_\theta = M_\theta/\frac{1}{2}\rho\ell^3 U_o^2$
$X'_u = X_u/\frac{1}{2}\rho\ell^2 U_o$	$M'_{\dot{u}} = M_{\dot{u}}/\frac{1}{2}\rho\ell^4$	$M'_{\delta_s} = M_{\delta_s}/\frac{1}{2}\rho\ell^3 U_o^2$
$X'_{\dot{w}} = X_{\dot{w}}/\frac{1}{2}\rho\ell^3$	$M'_u = M_u/\frac{1}{2}\rho\ell^3 U_o$	$I'_y = I_y/\frac{1}{2}\rho\ell^5$

The right-hand side of Equations (20) are the control forces and represent the means by which either a human operator, an autopilot, or a disturbance input can control the motion of the vehicle. The control surface and disturbance inputs are the forcing functions which determine the resultant motion of the vehicle. Since the vehicle equations of motion are linear equations, the principle of superposition may be used to obtain a solution. For example, the response to simultaneous application of sternplane and bowplane deflections can be determined by calculating the response to each of these deflections separately and then adding together the results to arrive at a complete solution.

In this report, only the longitudinal response to sternplane deflections and the lateral response to rudder deflections are given detailed analysis. It should be emphasized, however, that the mathematical techniques of solution for other control inputs are identical.

Applying the Laplace transform to Equations (20) yields

$$[(m - X_{\dot{u}})s - X_u]u + [- X_{\dot{w}}s - X_w]w + [- X_{\dot{q}}s^2 - X_q s - X_\theta]\theta = X_{\delta_s}\delta_s$$

$$[- Z_{\dot{u}}s - Z_u]u + [(m - Z_{\dot{w}})s - Z_w]w + [- Z_{\dot{q}}s^2 - (Z_q + mU_o)s - Z_\theta]\theta = Z_{\delta_s}\delta_s$$

$$[- M_{\dot{u}}s - M_u]u + [- M_{\dot{w}}s - M_w]w + [(I_y - M_{\dot{q}})s^2 - M_q s - M_\theta]\theta = M_{\delta_s}\delta_s \quad (21)$$

Equations (21) are nondimensionalized by dividing the force equations by $\frac{1}{2}\rho \ell^2 U_o^2$ and the moment equations by $\frac{1}{2}\rho \ell^3 U_o^2$. Dividing the first bracket in the X-force equations by $\frac{1}{2}\rho \ell^2 U_o^2$ yields

$$\frac{[(m - X_{\dot{u}})s - X_u]u}{\frac{1}{2}\rho \ell^2 U_o^2} = [(\frac{m}{\frac{1}{2}\rho \ell^3} - \frac{X_{\dot{u}}}{\frac{1}{2}\rho \ell^3})\frac{s}{U_o/\ell} - \frac{X_u}{\frac{1}{2}\rho \ell^2 U_o}]\frac{u}{U_o}$$

$$= [(m' - X'_{\dot{u}})s' - X'_u]u'$$

Likewise, for the second and third brackets and the right-hand side we obtain:

$$\frac{[- X_{\dot{w}}s - X_w]w}{\frac{1}{2}\rho \ell^2 U_o^2} = [- \frac{X_{\dot{w}}}{\frac{1}{2}\rho \ell^3}\frac{s}{U_o/\ell} - \frac{X_w}{\frac{1}{2}\rho \ell^2 U_o}]\frac{w}{U_o}$$

$$= [- X'_{\dot{w}}s' - X'_w]w'$$

$$\frac{[- X_{\dot{q}}s^2 - X_q s - X_\theta]\theta}{\frac{1}{2}\rho \ell^2 U_o^2} = [- \frac{X_{\dot{q}}}{\frac{1}{2}\rho \ell^4}\frac{s^2}{U_o^2/\ell^2} - \frac{X_q}{\frac{1}{2}\rho \ell^3 U_o}\frac{s}{U_o/\ell} - \frac{X_\theta}{\frac{1}{2}\rho \ell^2 U_o^2}]\theta$$

$$= [- X'_{\dot{q}}s'^2 - X'_q s' - X'_\theta]\theta$$

Finally for the right-hand side of the equation

$$\frac{X_{\delta_s}\delta_s}{\frac{1}{2}\rho \ell^2 U_o^2} = X'_{\delta_s}\delta_s .$$

Therefore, the nondimensional X-force equation becomes

$$[(m' - X'_{\dot{u}})s' - X'_u]u' + [- X'_{\dot{w}}s' - X'_w]w' + [-X'_{\dot{q}}s'^2 - X'_q s' - X'_\theta]\theta = X'_{\delta_s}\delta_s$$

Similarly, the remaining longitudinal equations can be placed in nondimensional form to yield:

$$[- Z'_{\dot{u}}s' - Z'_u]u' + [(m' - Z'_{\dot{w}})s' - Z'_w]w'$$
$$+ [- Z'_{\dot{q}}s'^2 - (Z'_q + m')s' - Z'_\theta]\theta = Z'_{\delta_s}\delta_s$$

$$[-M'_{\dot u}s' - M'_u]u' + [-M'_{\dot w}s' - M'_w]w'$$

$$+ [(I'_y - M'_{\dot q})s'^2 - M'_q s' - M'_\theta]\theta = M'_{\delta_s}\delta_s$$

These equations are shown in Table 2 for reference purposes. The equations predict the longitudinal small perturbation motions of self-propelled vehicles.

TABLE 2

NONDIMENSIONAL LONGITUDINAL EQUATIONS OF MOTION

$$[(m' - X'_{\dot u})s' - X'_u]u' - [X'_{\dot w}s' + X'_w]w' - [X'_{\dot q}s'^2 + X'_q s' + X'_\theta]\theta = X'_{\delta_s}\delta_s$$

$$[-Z'_{\dot u}s' - Z'_u]u' + [(m' - Z'_{\dot w})s' - Z'_w]w'$$

$$- [Z'_{\dot q}s'^2 + (Z'_q + m')s' + Z'_\theta]\theta = Z'_{\delta_s}\delta_s$$

$$[-M'_{\dot u}s' - M'_u]u' - [M'_{\dot w}s' + M'_w]w' + [(I'_y - M'_{\dot q})s'^2 - M'_q s' - M'_\theta]\theta = M'_{\delta_s}\delta_s$$

The transfer functions, for a given input, are obtained by solving the transformed simultaneous equations of motion for the output variable of interest with all other inputs considered to be zero. For example, using determinants, we can directly write the pitch attitude to control input transfer functions as

$$\frac{\theta(s)}{\delta_s(s)} = \frac{\begin{vmatrix} (m' - X'_{\dot u})s' - X'_u & -X'_{\dot w}s' - X'_w & X'_{\delta_s} \\ -Z'_{\dot u}s' - Z'_u & (m' - Z'_{\dot w})s' - Z'_w & Z'_{\delta_s} \\ -M'_{\dot u}s' - M'_u & -M'_{w}s' - M'_w & M'_{\delta_s} \end{vmatrix}}{\begin{vmatrix} (m' - X'_{\dot u})s' - X'_u & -X'_{\dot w}s' - X'_w & -X'_{\dot q}s'^2 - X'_q s' - X'_\theta) \\ -Z'_{\dot u}s' - Z'_u & (m' - Z'_{\dot w})s' - Z'_w & -Z_{\dot q}s'^2 - (Z'_q + m')s' + -Z'_\theta \\ -M'_{\dot u}s' - M'_u & -M'_{\dot w}s' - M'_w & (I'_y - M'_{\dot q})s'^2 - M'_q s' - M'_\theta \end{vmatrix}}$$

By expanding the determinants, the transfer function can be expressed as the ratio of a numerator polynomial in s over a denominator polynomial. The denominator polynomial $\Delta(s)$, is common to all the longitudinal transfer functions and its factors determine the frequency and damping, or time constants, of the individual modes of motion. The numerator polynomials depend on the output quantity of interest. The general polynomial forms of the primary longitudinal transfer functions are given in Table 3. The expressions for the various A, B, C, etc. coefficients of the equations in Table 3 is given in Appendix A in terms of hydrodynamic coefficients.

TABLE 3

LONGITUDINAL TRANSFER FUNCTIONS

$$\theta/\delta_s = \frac{N^{\theta}_{\delta_s}}{\Delta_{Long}} = \frac{A_\theta s'^2 + B_\theta s' + C_\theta}{As'^4 + Bs'^3 + Cs'^2 + Ds' + E}$$

$$w'/\delta_s = \frac{N^{W}_{\delta_s}}{\Delta_{Long}} = \frac{A_w s'^3 + B_w s'^2 + C_w s' + D_w}{As'^4 + Bs'^3 + Cs'^2 + Ds' + E}$$

$$u'/\delta_s = \frac{N^{U}_{\delta_s}}{\Delta_{Long}} = \frac{A_u s'^3 + B_u s'^2 + C_u s' + D_u}{As'^4 + Bs'^3 + Cs'^2 + Ds' + E}$$

Lateral Transfer Functions

The lateral transfer functions are derived in a manner similar to the longitudinal transfer functions. Substituting Equations (7) into Equations (19) yields

$$m\dot{v} + mU_o\dot{\psi} - Y_{\dot{v}}\dot{v} - Y_v v - Y_{\dot{p}}\ddot{\phi} - Y_p\dot{\phi} - Y_\phi\phi - Y_{\dot{r}}\ddot{\Psi} - Y_r\dot{\psi} = Y_{\delta_R}\,\delta_R$$

$$I_x\ddot{\phi} - I_{xz}\ddot{\Psi} - K_{\dot{v}}\dot{v} - K_v v - K_{\dot{p}}\ddot{\phi} - K_p\dot{\phi} - K_\phi\phi - K_{\dot{r}}\ddot{\Psi} - K_r\dot{\psi} = K_{\delta_R}\,\delta_R \qquad (25)$$

$$I_z\ddot{\Psi} - I_{xz}\ddot{\phi} - N_{\dot{v}}\dot{v} - N_v v - N_{\dot{p}}\ddot{\phi} - N_p\dot{\phi} - N_\phi\phi - N_{\dot{r}}\ddot{\Psi} - N_r\dot{\psi} = N_{\delta_R}\,\delta_R$$

The nondimensional, lateral equations of motion are obtained by applying the Laplace transform to Equations (25) and dividing the force equations by $\frac{1}{2}\rho\ell^3U_o^2$ and the moment equations by $\frac{1}{2}\rho\ell^2U_o^2$. Table 4 shows the lateral equations in terms of β, ϕ, and ψ where $\beta \equiv v/U_o$.

The lateral transfer functions are found from the equations of motion by expanding the numerator and denominator determinants as previously noted. The general polynomial forms of the primary lateral transfer functions are given in Table 5. Expressions for the coefficients A, B, C, etc. are given in Appendix B in terms of hydrodynamic coefficients.

TABLE 4

NONDIMENSIONAL LATERAL EQUATIONS OF MOTION

$$[(m' - Y'_{\dot{v}})s' - Y'_v]\beta + [- Y'_{\dot{p}}s'^2 - Y'_p s' - Y'_\phi]\phi$$

$$+ [- Y'_{\dot{r}}s'^2 + (m' - Y'_r)s']\psi = Y'_{\delta_R}\delta_R$$

$$[- K'_{\dot{v}}s' - K'_v]\beta + [(I'_x - K'_{\dot{p}})s'^2 - K'_p s' - K'_\phi]\phi$$

$$+ [(- I'_{xz} - K'_{\dot{r}})s'^2 - K'_r s']\psi = K'_{\delta_R}\delta_R$$

$$[- N'_{\dot{v}}s' - N'_v]\beta + [(- I'_{xz} - N'_{\dot{p}})s'^2 - N'_p s' - N'_\phi]\phi$$

$$+ [(I'_z - N'_{\dot{r}})s'^2 - N'_r s']\psi = N'_{\delta_R}\delta_R$$

TABLE 5

LATERAL TRANSFER FUNCTIONS

$$\beta/\delta_R = \frac{N^{\beta}_{\delta_R}}{\Delta_{Lat}} = \frac{s'(A_{\beta}s'^3 + B_{\beta}s'^2 + C_{\beta}s' + D_{\beta})}{s'(As'^4 + Bs'^3 + Cs'^2 + Ds' + E)}$$

$$\phi/\delta_R = \frac{N^{\phi}_{\delta_R}}{\Delta_{Lat}} = \frac{s'(A_{\phi}s'^2 + B_{\phi}s' + C_{\phi})}{s'(As'^4 + Bs'^3 + Cs'^2 + Ds' + E)}$$

$$\psi/\delta_R = \frac{N^{\psi}_{\delta_R}}{\Delta_{Lat}} = \frac{A_{\psi}s'^3 + B_{\psi}s'^2 + C_{\psi}s' + D_{\psi}}{s'(As'^4 + Bs'^3 + Cs'^2 + Ds' + E)}$$

EXAMPLE

To illustrate the use of the equations of motion and transfer functions, the dynamic characteristics of a typical vehicle are evaluated. The vehicle length is 49.33 feet while the velocity is 10.13 feet per second. The hydrodynamic coefficients of the vehicle are given in Table 6.

The transfer functions in Tables 3 and 5 are evaluated by substituting the numerical values of the hydrodynamic coefficients from Table 6 into the expressions in Appendix A and B. The numerical values for each transfer function coefficient are given in Table 7. These are the nondimensional transfer function coefficients. To obtain the dimensional form of the transfer function, substitute $s' = s\,\ell/U_o$ into the equations in Tables 3 and 5. For the velocity transfer functions u'/δ_s, v'/δ_R, and w'/δ_s, the left side of the equations in Tables 3 and 5 must also be multiplied by U_o to convert to the dimensional form. For illustrative purposes the nondimensional transfer functions θ/δ_s and v'/δ_R will be converted to the dimensional form. The nondimensional θ/δ_s transfer function,

$$\theta/\delta_s = \frac{1.10^{-4}(.3291\ s'^2 + .4746\ s' + .1361)}{-\ 1.10^{-4}(.0902s'^4 + .3960s'^3 + .3288s'^2 + .1028s' + .0115)}$$

is converted to the dimensional form given below by the substitution $s' = s\,\ell/U_o$.

(Text Continued on Page 23)

TABLE 6

NONDIMENSIONAL HYDRODYNAMIC COEFFICIENTS

$X'_u = -0.015020$	$Y'_v = -0.082369$
$X'_{\dot{u}} = -0.001623$	$Y'_r = 0.026955$
$X'_\theta = -0.000086$	$Y'_\phi = 0.000086$
$X'_{\delta_s} = -0.002770$	$Y'_{\dot{v}} = -0.035545$
$Z'_w = -0.050138$	$Y'_{\dot{p}} = -0.000190$
$Z'_q = -0.017455$	$Y'_{\dot{r}} = 0.000400$
$Z'_{\dot{w}} = -0.031545$	$Y'_{\delta_R} = 0.026262$
$Z'_{\dot{q}} = -0.000130$	$K'_v = 0.002579$
$Z'_{\delta_s} = -0.027695$	$K'_p = -0.000067$
$M'_w = 0.009550$	$K'_r = -0.000280$
$M'_\theta = -0.001530$	$K'_\phi = -0.001530$
$M'_q = -0.011310$	$K'_{\dot{v}} = 0.000190$
$M'_{\dot{w}} = -0.000146$	$K'_{\dot{p}} = -0.000092$
$M'_{\dot{q}} = -0.001573$	$K'_{\dot{r}} = -0.000042$
$M'_{\delta_s} = -0.012797$	$K'_{\delta_R} = 0.000334$
$m' = 0.036397$	$N'_v = -0.018264$
$I'_x = 0.000047$	$N'_r = -0.012497$
$I'_y = 0.001917$	$N'_{\dot{v}} = -0.000182$
$I'_z = 0.001547$	$N'_{\dot{r}} = -0.001531$
$\ell = 49.33$ feet	$N'_{\delta_R} = -0.012663$
$U_o = 10.13$ ft/sec	

TABLE 7

NONDIMENSIONAL TRANSFER FUNCTION COEFFICIENTS

	A	B	C	D	E
Δ_{Long}	-0.00000902	-0.00003960	-0.00003288	-0.00001028	-0.00000115
$N^{u'}_{\delta_s}$	0.00000066	0.00000262	0.00000128	0.00000136	
$N^{w'}_{\delta_s}$	0.00000361	0.00002256	0.00000996	0.00000064	
$N^{\theta}_{\delta_s}$	0.00003291	0.00004746	0.00001361		
Δ_{Lat}	-0.00000003	-0.00000018	-0.00000054	-0.00000183	-0.00000131
$N^{v'}_{\delta_R}$	-0.00000001	-0.00000007	-0.00000015	-0.00000069	
$N^{\psi}_{\delta_R}$	0.00000013	0.00000028	0.00000149	0.00000233	
$N^{\phi}_{\delta_R}$	-0.00000013	-0.00000099	-0.00000187		

$$\theta/\delta_s = \frac{-\ 1.10^{-3}[.7808\ s^2 + .2311\ s + .0136]}{1.10^{-3}[5.0769\ s^4 + 4.5761\ s^3 + .7799\ s^2 + .0501\ s + .0012]}$$

Likewise for the v'/δ_R transfer function,

$$v'/\delta_R = \frac{-\ 1.10^{-5}[.001\ s'^4 + .007\ s'^3 + .015\ s'^2 + .069\ s']}{-\ 1.10^{-5}[.003\ s'^5 + .018\ s'^4 + .054\ s'^3 + .183\ s'^2 + .131\ s']}$$

is converted to dimensional form by the substitutions $s' = s\ \ell/U_o$ and $v' = v/U_o$. Thus, the dimensional transfer function is

$$v/\delta_R = \frac{.5829\ s^4 + .7698\ s^3 + .3510\ s^2 + .3383\ s}{.8460\ s^5 + 1.003\ s^4 + .6272\ s^3 + .4352\ s^2 + .0637\ s}$$

The dimensional transfer functions are given in Table 8 in the polynomial and factored form.

(Text Continued on Page 25)

TABLE 8

DIMENSIONAL TRANSFER FUNCTIONS

$$u/\delta_s = \frac{1.0.10^{-3}(0.7688s^3 + 0.6306s^2 + 0.0633s + 0.0014)}{-\ 1.0.10^{-2}(0.5077s^4 + 0.4576s^3 + 0.0780s^2 + 0.0050s + 0.0001)}$$

$$= \frac{-\ 0.1514(s + 0.0302)(s + 0.0826)(s + 0.7074)}{[(s + .0592)^2 + (0.0219)^2](s + 0.0811)(s + 0.7018)}$$

$$w/\delta_s = \frac{1.0.10^{-2}(0.4228s^3 + 0.5420s^2 + 0.0491s + 0.0006)}{-\ 1.0.10^{-2}(0.5077s^4 + 0.4576s^3 + 0.0780s^2 + 0.0050s + 0.0001)}$$

$$= \frac{-\ 0.8328(s + 0.0158)(s + 0.0811)(s + 1.1849)}{[(s + .0592)^2 + (0.0219)^2](s + 0.0811)(s + 0.7018)}$$

$$\theta/\delta_s = \frac{1.0.10^{-3}(0.7808s^2 + 0.2311s + 0.0136)}{-\ 1.0.10^{-2}(0.5077s^4 + 0.4576s^3 + 0.0780s^2 + 0.0050s + 0.0001)}$$

$$= \frac{-\ 0.1538(s + 0.0811)\ (s + 0.2150)}{[(s + .0592)^2 + (0.0219)^2](s + 0.0811)(s + 0.7018)}$$

$$v/\delta_R = \frac{-\ 1.0.10^{-4}(0.5829s^4 + 0.7698s^3 + 0.3510s^2 + 0.33835)}{-\ 1.0.10^{-4}(0.8459s^5 + 1.0033s^4 + 0.6272s^3 + 0.4352s^2 + 0.0637s)}$$

$$= \frac{0.6890(s)[(s + .0516)^2 + (0.6885)^2](s + 1.2176)}{s[(s + .0543)^2 + (0.6768)^2](s + 0.1826)(s + 0.8949)}$$

$$\psi/\delta_R = \frac{-\ 1.0.10^{-4}(0.1460s^3 + 0.0658s^2 + 0.0725s + 0.0233)}{-\ 1.0.10^{-4}(0.8459s^5 + 1.0033s^4 + 0.6272s^3 + 0.4352s^2 + 0.0637s)}$$

$$= \frac{0.1725[(s + .0522)^2 + (0.6767)^2](s + 0.3462)}{s[(s + .0543)^2 + (0.6768)^2](s + 0.1826)(s + 0.8949)}$$

$$\phi/\delta_R = \frac{-\ 1.0.10^{-4}(0.1461s^3 + 0.2340s^2 + 0.0910s)}{-\ 1.0.10^{-4}(0.8459s^5 + 1.0033s^4 + 0.6272s^3 + 0.4352s^2 + 0.0637s)}$$

$$= \frac{0.1727(s)(s + 0.9378)(s + 0.6638)}{s[(s + .0543)^2 + (0.6768)^2](s + 0.1826)(s + 0.8949)}$$

The characteristic equation for longitudinal motions has four roots. The second order mode is low frequency (0.06 rad/sec) and very highly damped (ζ=0.94). In the forward velocity (u) transfer functions, the zeros at 0.0826 and 0.7074 very nearly cancel the poles at 0.0811 and 0.7018. Thus the perturbations in forward velocity will be represented by a second order response with one overshoot and no undershoot, with the 0.95 value for settling time equal to 50.7 secs. The response in vertical velocity (w) and pitch (θ) are similar to the u-response, the major difference being the value of the numerator zero.

The characteristic equation for the lateral motions has five roots. First, there is a free s in the denominator. Then, there is a second order mode which is of medium frequency (0.68 rad/sec) and very lightly damped (ζ=0.08). For the side velocity (v) transfer function, the free s is canceled in the numerator and the second order mode nearly cancels the second order mode in the numerator. Thus, perturbation in side velocity will have a first order response mode. The characteristic mode will be the root at 0.1826 with a 0.95 value for settling time equal to 16.4 seconds.

For the yaw transfer function, the second order mode again is nearly canceled but the free s in the denominator remains. Thus, the response will be characterized by a first order integrator. For the roll transfer functions, the free s is canceled in the numerator and the second order mode is the characteristic response mode. Thus, the motions in roll will be oscillatory, with a period of 9.25 seconds and lightly damped.

Figures 3 and 4 show the time responses of the vehicle for a one-degree step in the control surface. These time responses were computed on a digital computer using the full transfer functions as shown in Table 8.

SUMMARY

The linear, small-perturbation equations of motion have been derived by equating the forces and moments acting on the vehicle to its reactions. Appropriate assumptions are introduced to separate the longitudinal motions from the lateral motions. The Laplace transform method of solution is used to solve for the independent variables. Expansion of the transfer functions polynomial coefficients is completed and the results are presented in the appendices for reference. An example problem is provided to demonstrate the use of the transfer functions and to present typical values for the characteristic vehicle motions.

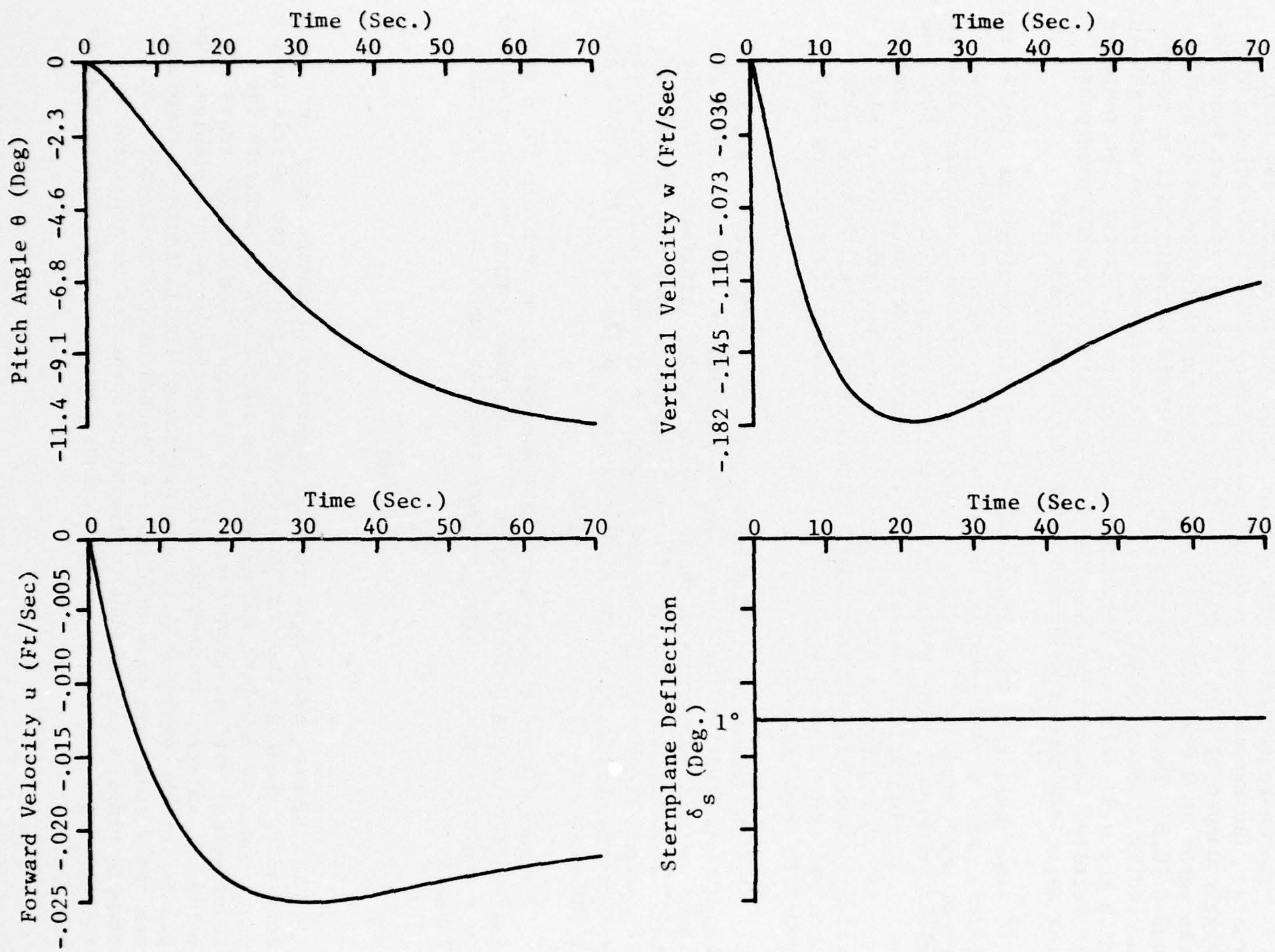

FIGURE 3. TIME RESPONSES FOR LONGITUDINAL MOTIONS

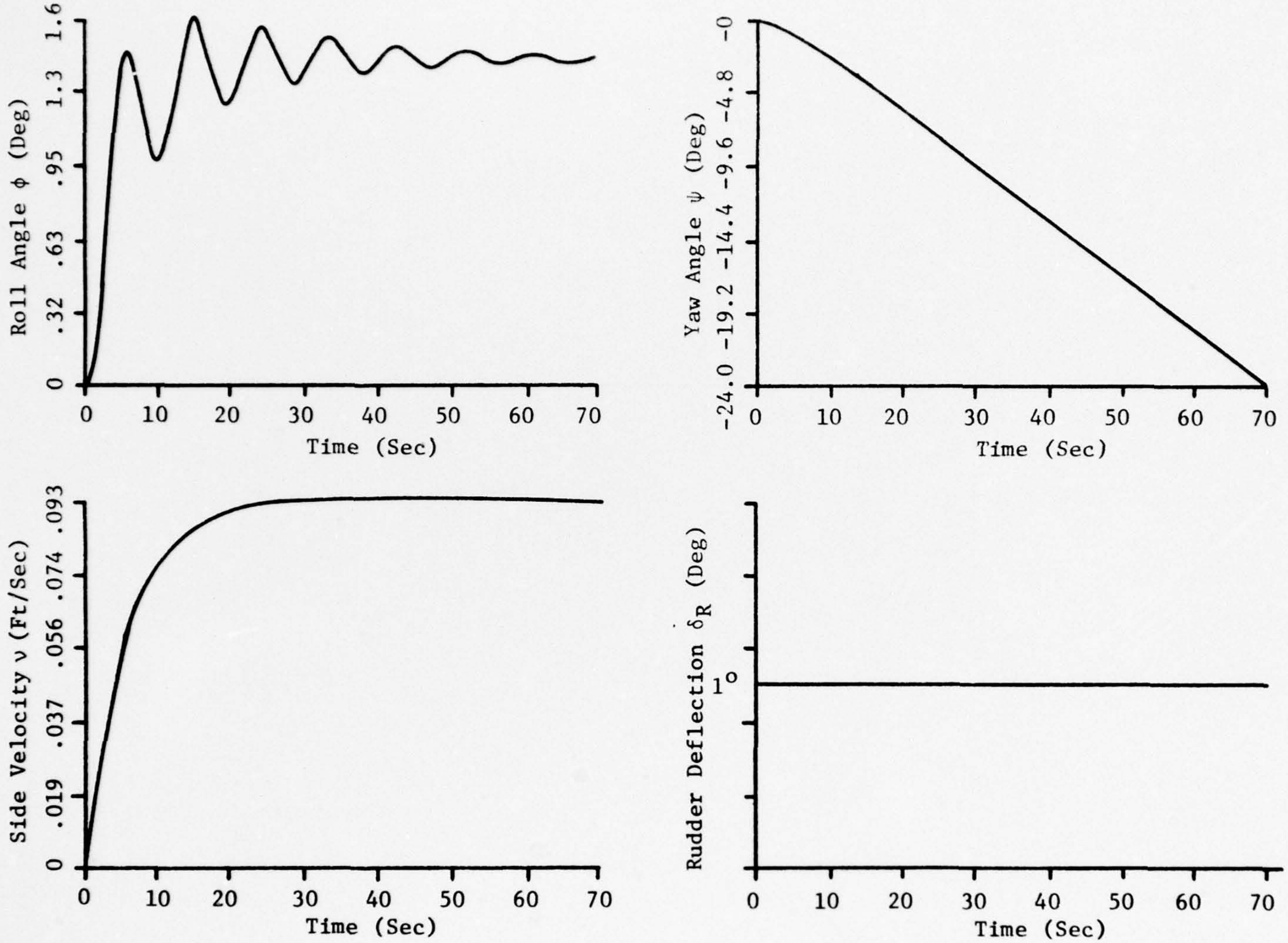

FIGURE 4. TIME RESPONSES FOR LATERAL MOTIONS

APPENDIX A

EXPRESSIONS FOR THE LONGITUDINAL TRANSFER FUNCTION COEFFICIENTS

The longitudinal characteristic equation is

$$\Delta_{Long} = As'^4 + Bs'^3 + Cs'^2 + Ds' + E$$

where

$$\begin{aligned}
A = {} & (m' - X'_{\dot u})(m' - Z'_{\dot w})(I'_y - M'_{\dot q}) - X'_{\dot w}M'_{\dot u}Z'_{\dot q} - Z'_{\dot u}M'_{\dot w}X'_{\dot q} \\
& - (m' - Z'_{\dot w})M'_{\dot u}X'_{\dot q} - Z'_{\dot u}X'_{\dot w}(I'_y - M'_{\dot q}) - M'_{\dot w}(m' - X'_{\dot u})Z'_{\dot q} .
\end{aligned}$$

$$\begin{aligned}
B = {} & -(m' - X'_{\dot u})(m' - Z'_{\dot w})M'_q - Z'_w(m' - X'_{\dot u})(I'_y - M'_{\dot q}) \\
& - X'_u(m' - Z'_{\dot w})(I'_y - M'_{\dot q}) - X'_{\dot w}M'_{\dot u}(Z'_q + m') - M'_uX'_{\dot w}Z'_{\dot q} - M'_{\dot u}X'_wZ'_{\dot q} \\
& - Z'_{\dot u}M'_{\dot w}X'_q - Z'_{\dot u}M'_wZ'_{\dot q} - Z'_uM'_{\dot w}X'_{\dot q} - (m' - Z'_{\dot w})M'_{\dot u}X'_q - M'_u(m' - Z'_{\dot w})X'_{\dot q} \\
& + M'_{\dot u}Z'_wX'_{\dot q} + Z'_{\dot u}X'_{\dot w}M'_q - Z'_uX'_{\dot w}(I'_y - M'_{\dot q}) - Z'_{\dot u}X'_w(I'_y - M'_{\dot q}) \\
& - M'_{\dot w}(m' - X'_{\dot u})(Z'_q + m') - M'_w(m' - X'_{\dot u})Z'_{\dot q} + M'_{\dot w}X'_uZ'_{\dot q} .
\end{aligned}$$

$$\begin{aligned}
C = {} & -(m' - X'_{\dot u})(m' - Z'_{\dot w})M'_\theta + Z'_w(m' - X'_{\dot u})M'_q + X'_u(m' - Z'_{\dot w})M'_q \\
& + Z'_wX'_u(I'_y - M'_{\dot q}) - X'_{\dot w}M'_{\dot u}Z'_\theta - M'_uX'_{\dot w}(Z'_q + m') - M'_{\dot u}X'_w(Z'_q + m') \\
& - Z'_{\dot u}M'_{\dot w}X'_\theta - Z'_{\dot u}M'_wX'_q - Z'_uM'_{\dot w}X'_q - Z'_uM'_wX'_{\dot q} - (m' - Z'_{\dot w})M'_{\dot u}X'_\theta \\
& - M'_u(m' - Z'_{\dot w})X'_q + M'_{\dot u}Z'_wX'_q + Z'_wM'_uX'_{\dot q} + Z'_{\dot u}X'_{\dot w}M'_\theta + Z'_uX'_{\dot w}M'_q + Z'_{\dot u}X'_wM'_q \\
& - X'_wZ'_u(I'_y - M'_{\dot q}) - M'_{\dot w}(m' - X'_{\dot u})Z'_\theta - M'_w(m' - X'_{\dot u})(Z'_q + m') \\
& + M'_{\dot w}X'_u(Z'_q + m') + X'_uM'_wZ'_{\dot q} - M'_uX'_wZ'_{\dot q} .
\end{aligned}$$

$$D = Z'_w M'_\theta (m' - X'_{\dot u}) + X'_u (m' - Z'_{\dot w}) M'_\theta - Z'_w X'_u M'_q - M'_u X'_{\dot w} Z'_\theta - M'_{\dot u} X'_w Z'_\theta$$

$$- M'_u X'_w (Z'_q + m') - Z'_{\dot u} M'_w X'_\theta - Z'_u M'_{\dot w} X'_\theta - Z'_u M'_w X'_q - M'_u (m' - Z'_{\dot w}) X'_\theta$$

$$+ M'_{\dot u} Z'_w X'_\theta + Z'_w M'_u X'_q + Z'_u X'_{\dot w} M'_\theta + Z'_{\dot u} X'_w M'_\theta + X'_w Z'_u M'_q - M'_w (m' - X'_{\dot u}) Z'_\theta$$

$$+ M'_{\dot w} X'_u Z'_\theta + M'_w X'_u (Z'_q + m') \ .$$

$$E = - Z'_w X'_u M'_\theta - M'_u X'_w Z'_\theta - Z'_u M'_w X'_\theta + Z'_w M'_u X'_\theta + X'_w Z'_u M'_\theta + X'_u M'_w Z'_\theta \ .$$

The pitch response transfer function is

$$\theta/\delta_s = \frac{N^\theta_{\delta_s}}{\Delta_{Long}} = \frac{A_\theta s'^2 + B_\theta s' + C_\theta}{\Delta_{Long}}$$

where

$$A_\theta = M'_{\delta_e} (m' - X'_{\dot u})(m' - Z'_{\dot w}) + Z'_{\delta_e} X'_{\dot w} M'_{\dot u} + X'_{\delta_e} Z'_{\dot u} M'_{\dot w} - M'_{\delta_e} X'_{\dot w} Z'_{\dot u}$$

$$+ X'_{\delta_e} (m' - Z'_{\dot w}) M'_{\dot u} + Z'_{\delta_e} M'_{\dot w} (m' - X'_{\dot u}) \ .$$

$$B_\theta = - M'_{\delta_e} (m' - X'_{\dot u}) Z'_w - M'_{\delta_e} X'_u (m' - Z'_{\dot w}) + Z'_{\delta_e} X'_{\dot w} M'_u + Z'_{\delta_e} X'_w M'_{\dot u}$$

$$+ X'_{\delta_e} Z'_{\dot u} M'_w + X'_{\delta_e} Z'_u M'_{\dot w} - M'_{\delta_e} X'_{\dot w} Z'_u - M'_{\delta_e} X'_w Z'_{\dot u} + X'_{\delta_e} (m' - Z'_{\dot w}) M'_u$$

$$- X'_{\delta_e} Z'_w M'_{\dot u} - Z'_{\delta_e} M'_{\dot w} X'_u + Z'_{\delta_e} M'_w (m' - X'_{\dot u}) \ .$$

$$C_\theta = M'_{\delta_e} X'_u Z'_w + Z'_{\delta_e} X'_w M'_u + X'_{\delta_e} Z'_u M'_w - M'_{\delta_e} X'_w Z'_u - X'_{\delta_e} Z'_w M'_u - Z'_{\delta_e} M'_w X'_u \ .$$

The vertical velocity transfer function is

$$\frac{w'}{\delta_s} = \frac{N^w_{\delta_s}}{\Delta_{Long}} = \frac{A_w s'^3 + B_w s'^2 + C_w s' + D_w}{\Delta_{Long}}$$

where

$$A_w = Z'_{\delta_e}(m' - X'_{\dot{u}})(I'_y - M'_{\dot{q}}) + X'_{\delta_e} M'_{\dot{u}} Z'_{\dot{q}} + M'_{\delta_e} Z'_{\dot{u}} X'_{\dot{q}} - Z'_{\delta_e} M'_{\dot{u}} X'_{\dot{q}}$$

$$+ X'_{\delta_e} Z'_{\dot{u}}(I'_y - M'_{\dot{q}}) + M'_{\delta_e}(m' - X'_{\dot{u}}) Z'_{\dot{q}} .$$

$$B_w = - Z'_{\delta_e}(m' - X'_{\dot{u}}) M'_q - Z'_{\delta_e} X'_u (I'_y - M'_{\dot{q}}) + X'_{\delta_e} M'_u Z'_{\dot{q}} + X'_{\delta_e} M'_{\dot{u}}(Z'_q + m')$$

$$+ M'_{\delta_e} Z'_{\dot{u}} X'_q + M'_{\delta_e} Z'_u X'_{\dot{q}} - Z'_{\delta_e} M'_{\dot{u}} X'_q - Z'_{\delta_e} M'_u X'_{\dot{q}} - X'_{\delta_e} Z'_{\dot{u}} M'_q$$

$$+ X'_{\delta_e} Z'_u (I'_y - M'_{\dot{q}}) + M'_{\delta_e}(m' - X'_{\dot{u}})(Z'_q + m') - M'_{\delta_e} X'_u Z'_{\dot{q}} .$$

$$C_w = - Z'_{\delta_e}(m' - X'_{\dot{u}}) M'_\theta + Z'_{\delta_e} X'_u M'_q + X'_{\delta_e} M'_{\dot{u}} Z'_\theta + X'_{\delta_e} M'_u (Z'_q + m') + M'_{\delta_e} Z'_{\dot{u}} X'_\theta$$

$$+ M'_{\delta_e} Z'_u Z'_q - Z'_{\delta_e} M'_{\dot{u}} X'_\theta - Z'_{\delta_e} M'_u X'_q - X'_{\delta_e} Z'_{\dot{u}} M'_\theta - X'_{\delta_e} Z'_u M'_q + M'_{\delta_e}(m' - X'_{\dot{u}}) Z'_\theta$$

$$- M'_{\delta_e} X'_u (Z'_q + m').$$

$$D_w = Z'_{\delta_e} X'_u M'_\theta + X'_{\delta_e} M'_u Z'_\theta + M'_{\delta_e} Z'_u X'_\theta - Z'_{\delta_e} M'_u X'_\theta - X'_{\delta_e} Z'_u M'_\theta - M'_{\delta_e} X'_u Z'_\theta .$$

The forward speed transfer function is

$$\frac{u'}{\delta_s} = \frac{N^u_{\delta_s}}{\Delta_{Long}} = \frac{A_u s'^3 + B_u s'^2 + C_u s' + D_u}{\Delta_{Long}}$$

where

$$A_u = X'_{\delta_e}(m' - Z'_{\dot{w}})(I'_y - M'_{\dot{q}}) + M'_{\delta_e} X'_{\dot{w}} Z'_{\dot{q}} + Z'_{\delta_e} M'_{\dot{w}} X'_{\dot{q}} + M'_{\delta_e}(m' - Z'_{\dot{w}}) X'_{\dot{q}}$$

$$+ Z'_{\delta_e} X'_{\dot{w}}(I'_y - M'_{\dot{q}}) - X'_{\delta_e} M'_{\dot{w}} Z'_{\dot{q}} .$$

$$B_u = - X'_{\delta_e}(m' - Z'_{\dot{w}})M'_q - X'_{\delta_e}(I'_y - M'_{\dot{q}})Z'_w + M'_{\delta_e} X'_{\dot{w}}(Z'_q + m') + M'_{\delta_e} X'_w Z'_{\dot{q}}$$

$$+ Z'_{\delta_e} M'_{\dot{w}} X'_q + Z'_{\delta_e} M'_w X'_{\dot{q}} + M'_{\delta_e}(m' - Z'_{\dot{w}})X'_q - M'_{\delta_e} Z'_w X'_{\dot{q}} - Z'_{\delta_e} X'_{\dot{w}} M'_q$$

$$+ Z'_{\delta_e} X'_w (I'_y - M'_{\dot{q}}) - X'_{\delta_e} M'_{\dot{w}}(Z'_q + m') - X'_{\delta_e} M'_w Z'_{\dot{q}} \; .$$

$$C_u = - X'_{\delta_e}(m' - Z'_{\dot{w}})M'_\theta + X'_{\delta_e} Z'_w M'_q + M'_{\delta_e} X'_{\dot{w}} Z'_\theta + M'_{\delta_e} X'_w (Z'_q + m') + Z'_{\delta_e} M'_{\dot{w}} X'_\theta$$

$$+ Z'_{\delta_e} M'_w X'_q + M'_{\delta_e}(m' - Z'_{\dot{w}})X'_\theta - M'_{\delta_e} Z'_w X'_q - Z'_{\delta_e} X'_{\dot{w}} M'_\theta - Z'_{\delta_e} X'_w M'_q$$

$$- X'_{\delta_e} M'_{\dot{w}} Z'_\theta - X'_{\delta_e} M'_w (Z'_q + m') \; .$$

$$D_u = X'_{\delta_e} Z'_w M'_\theta + M'_{\delta_e} X'_w Z'_\theta + Z'_{\delta_e} M'_w X'_\theta - M'_{\delta_e} Z'_w X'_\theta - Z'_{\delta_e} X'_w M'_\theta - X'_{\delta_e} M'_w Z'_\theta \; .$$

APPENDIX B

EXPRESSIONS FOR THE LATERAL TRANSFER FUNCTION COEFFICIENTS

The lateral characteristic equation is

$$\Delta_{Lat} = s'(As'^4 + Bs'^3 + Cs'^2 + Ds' + E)$$

where

$$A = (m' - Y'_{\dot{v}})(I'_z - N'_{\dot{r}})(I'_x - K'_{\dot{p}}) + N'_{\dot{v}}(- I'_{xz} - K'_{\dot{r}})Y'_{\dot{p}}$$

$$+ Y'_{\dot{r}}K'_{\dot{v}}(- I'_{xz} - N'_{\dot{p}}) - N'_{\dot{v}}Y'_{\dot{r}}(I'_x - K'_{\dot{p}}) - K'_{\dot{v}}(I'_z - N'_{\dot{r}})Y'_{\dot{p}}$$

$$- (m' - Y'_{\dot{v}})(- I'_{xz} - N'_{\dot{p}})(- I'_{xz} - K'_{\dot{r}}) .$$

$$B = - (m' - Y'_{\dot{v}})(I'_z - N'_{\dot{r}})K'_p - N'_r(m' - Y'_{\dot{v}})(I'_x - K'_{\dot{p}})$$

$$- Y'_v(I'_z - N'_{\dot{r}})(I'_x - K'_{\dot{p}}) + N'_{\dot{v}}(- I'_{xz} - K'_{\dot{r}})Y'_p - N'_{\dot{v}}K'_rY'_{\dot{p}}$$

$$+ N'_v(- I'_{xz} - K'_{\dot{r}})Y'_{\dot{p}} - Y'_{\dot{r}}K'_{\dot{v}}N'_p + Y'_{\dot{r}}K'_v(- I'_{xz} - N'_{\dot{p}})$$

$$- (m' - Y'_r)K'_{\dot{v}}(- I'_{xz} - N'_{\dot{p}}) + N'_{\dot{v}}Y'_{\dot{r}}K'_p + N'_{\dot{v}}(m' - Y'_r)(I'_x - K'_{\dot{p}})$$

$$- N'_vY'_{\dot{r}}(I'_x - K'_{\dot{p}}) - K'_{\dot{v}}(I'_z - N'_{\dot{r}})Y'_p + K'_{\dot{v}}N'_rY'_{\dot{p}} - (I'_z - N'_{\dot{r}})K'_vY'_{\dot{p}}$$

$$+ (m' - Y'_{\dot{v}})(- I'_{xz} - K'_{\dot{r}})N'_p + (m' - Y'_{\dot{v}})K'_r(- I'_{xz} - N'_{\dot{p}})$$

$$+ Y'_v(- I'_{xz} - K'_{\dot{r}})(- I'_{xz} - N'_{\dot{p}}) .$$

$$C = N'_r(m' - Y'_{\dot v})K'_p - (m' - Y'_{\dot v})(I'_z - N'_{\dot r})K'_\phi + Y'_v(I'_z - N'_{\dot r})K'_p$$

$$+ N'_r Y'_v(I'_x - K'_{\dot p}) + N'_{\dot v}(- I'_{xz} - K'_{\dot r})Y'_\phi - N'_{\dot v}K'_r Y'_p + N'_v(- I'_{xz} - K'_{\dot r})Y'_p$$

$$- N'_v K'_r Y'_{\dot p} - Y'_{\dot r}K'_{\dot v}N'_\phi - Y'_{\dot r}K'_v N'_p + (m' - Y'_r)K'_{\dot v}N'_p$$

$$- (m' - Y'_r)K'_v(- I'_{xz} - N'_{\dot p}) + N'_{\dot v}Y'_{\dot r}K'_\phi - N'_{\dot v}(m' - Y'_r)K'_p + N'_v Y'_{\dot r}K'_p$$

$$+ N'_v(m' - Y'_r)(I'_x - K'_{\dot p}) - K'_{\dot v}(I'_z - N'_{\dot r})Y'_\phi + K'_{\dot v}N'_r Y'_p - (I'_z - N'_{\dot r})K'_v Y'_p$$

$$+ K'_v N'_r Y'_{\dot p} + (m' - Y'_{\dot v})(- I'_{xz} - K'_{\dot r})N'_\phi - (m' - Y'_{\dot v})K'_r N'_p$$

$$- Y'_v(- I'_{xz} - K'_{\dot r})N'_p - K'_r Y'_v(- I'_{xz} - N'_{\dot p}) .$$

$$D = N'_r(m' - Y'_{\dot v})K'_\phi + Y'_v(I'_z - N'_{\dot r})K'_\phi - N'_r Y'_v K'_p - N'_{\dot v}K'_r Y'_\phi$$

$$+ N'_v(- I'_{xz} - K'_{\dot r})Y'_\phi - N'_v K'_r Y'_p - Y'_{\dot r}K'_v N'_\phi + (m' - Y'_r)K'_{\dot v}N'_\phi$$

$$+ (m' - Y'_r)K'_v N'_p - N'_{\dot v}(m' - Y'_r)K'_\phi + N'_v Y'_{\dot r}K'_\phi - N'_v(m' - Y'_r)K'_p + K'_{\dot v}N'_r Y'_\phi$$

$$- (I'_z - N'_{\dot r})K'_v Y'_\phi + K'_v N'_r Y'_p - (m' - Y'_{\dot v})K'_r N'_\phi - Y'_v(- I'_{xz} - K'_{\dot r})N'_\phi$$

$$+ K'_r Y'_v N'_p .$$

$$E = - N'_r Y'_v K'_\phi - N'_v K'_r Y'_\phi + (m' - Y'_r)K'_v N'_\phi - N'_v(m' - Y'_r)K'_\phi + K'_v N'_r Y'_\phi + K'_r Y'_v N'_\phi .$$

The sideslip transfer function is

$$\frac{\beta}{\delta_R} = \frac{N^{\beta}_{\delta_R}}{\Delta_{Lat}} = \frac{s'(A_\beta s'^3 + B_\beta s'^2 + C_\beta s' + D_\beta)}{\Delta_{Lat}}$$

where

$$A_\beta = Y'_{\delta_r}(I'_z - N'_{\dot r})(I'_x - K'_{\dot p}) - N'_{\delta_r}(-I'_{xz} - K'_{\dot r})Y'_{\dot p} - K'_{\delta_r}Y'_{\dot r}(-I'_{xz} - N'_{\dot p})$$

$$+ N'_{\delta_r}Y'_{\dot r}(I'_x - K'_{\dot p}) + K'_{\delta_r}(I'_z - N'_{\dot r})Y'_{\dot p} - Y'_{\delta_r}(-I'_{xz} - K'_{\dot r})(-I'_{xz} - N'_{\dot p}) \, .$$

$$B_\beta = -Y'_{\delta_r}(I'_z - N'_{\dot r})K'_p - Y'_{\delta_r}N'_r(I'_x - K'_{\dot p}) - N'_{\delta_r}(-I'_{xz} - K'_{\dot r})Y'_p + N'_{\delta_r}K'_rY'_{\dot p}$$

$$+ K'_{\delta_r}Y'_{\dot r}N'_p + K'_{\delta_r}(m' - Y'_r)(-I'_{xz} - N'_{\dot p}) - N'_{\delta_r}Y'_{\dot r}K'_p$$

$$- (m' - Y'_r)N'_{\delta_r}(I'_x - K'_{\dot p}) + K'_{\delta_r}(I'_z - N'_{\dot r})Y'_p - K'_{\delta_r}N'_rY'_{\dot p}$$

$$+ Y'_{\delta_r}(-I'_{xz} - K'_{\dot r})N'_p + Y'_{\delta_r}K'_r(-I'_{xz} - N'_{\dot p}) \, .$$

$$C_\beta = -Y'_{\delta_r}(I'_z - N'_{\dot r})K'_\phi + Y'_{\delta_r}N'_rK'_p - N'_{\delta_r}(-I'_{xz} - K'_{\dot r})Y'_\phi + N'_{\delta_r}K'_rY'_p$$

$$+ K'_{\delta_r}Y'_{\dot r}N'_\phi - K'_{\delta_r}(m' - Y'_r)N'_p - N'_{\delta_r}Y'_{\dot r}K'_\phi + (m' - Y'_r)N'_{\delta_r}K'_p$$

$$+ K'_{\delta_r}(I'_z - N'_{\dot r})Y'_\phi - N'_rK'_{\delta_r}Y'_p + Y'_{\delta_r}(-I'_{xz} - K'_{\dot r})N'_\phi - Y'_{\delta_r}K'_rN'_p \, .$$

$$D_\beta = Y'_{\delta_r}N'_rK'_\phi + N'_{\delta_r}K'_rY'_\phi - K'_{\delta_r}(m' - Y'_r)N'_\phi + (m' - Y'_r)N'_{\delta_r}K'_\phi - N'_rK'_{\delta_r}Y'_\phi$$

$$- Y'_{\delta_r}K'_rN'_\phi \, .$$

The roll transfer function is

$$\frac{\phi}{\delta_R} = \frac{N^{\phi}_{\delta_R}}{\Delta_{Lat}} = \frac{s'(A_\phi s'^2 + B_\phi s' + C_\phi)}{\Delta_{Lat}}$$

where

$$A_\phi = K'_{\delta_r}(m' - Y'_{\dot{v}})(I'_z - N'_{\dot{r}}) - Y'_{\delta_r} N'_{\dot{v}}(- I'_{xz} - K'_{\dot{r}}) + N'_{\delta_r} K'_{\dot{v}} Y'_{\dot{r}}$$

$$- K'_{\delta_r} N'_{\dot{v}} Y'_{\dot{r}} + Y'_{\delta_r} K'_{\dot{v}}(I'_z - N'_{\dot{r}}) - N'_{\delta_r}(m' - Y'_{\dot{v}})(- I'_{xz} - K'_{\dot{r}}) .$$

$$B_\phi = - K'_{\delta_r}(m' - Y'_{\dot{v}})N'_r - K'_{\delta_r} Y'_v(I'_z - N'_{\dot{r}}) + Y'_{\delta_r} N'_{\dot{v}} K'_r - Y'_{\delta_r} N'_v(- I'_{xz} - K'_{\dot{r}})$$

$$- N'_{\delta_r} K'_{\dot{v}}(m' - Y'_r) + N'_{\delta_r} K'_v Y'_{\dot{r}} + K'_{\delta_r} N'_{\dot{v}}(m' - Y'_r) - K'_{\delta_r} N'_v Y'_{\dot{r}} - Y'_{\delta_r} K'_{\dot{v}} N'_r$$

$$+ Y'_{\delta_r} K'_v(I'_z - N'_{\dot{r}}) + N'_{\delta_r}(m' - Y'_{\dot{v}})K'_r + N'_{\delta_r} Y'_v(- I'_{xz} - K'_{\dot{r}}) .$$

$$C_\phi = K'_{\delta_r} Y'_v N'_r + Y'_{\delta_r} N'_v K'_r - N'_{\delta_r} K'_v(m' - Y'_r) + K'_{\delta_r} N'_v(m' - Y'_r) - Y'_{\delta_r} K'_v N'_r$$

$$- N'_{\delta_r} Y'_v K'_r .$$

The yaw transfer function is

$$\frac{\psi}{\delta_R} = \frac{N^{\psi}_{\delta_R}}{\Delta_{Lat}} = \frac{A_\psi s'^3 + B_\psi s'^2 + C_\psi s' + D_\psi}{\Delta_{Lat}}$$

where

$$A_\psi = N'_{\delta_r}(m' - Y'_{\dot{v}})(I'_x - K'_{\dot{p}}) + K'_{\delta_r} N'_{\dot{v}} Y'_{\dot{p}} - Y'_{\delta_r} K'_{\dot{v}}(- I'_{xz} - N'_{\dot{p}})$$

$$+ Y'_{\delta_r} N'_{\dot{v}}(I'_x - K'_{\dot{p}}) - N'_{\delta_r} K'_{\dot{v}} Y'_{\dot{p}} - K'_{\delta_r}(m' - Y'_{\dot{v}})(- I'_{xz} - N'_{\dot{p}}) .$$

$$B_\psi = - N'_{\delta_r}(m' - Y'_{\dot{v}})K'_p - N'_{\delta_r} Y'_v(I'_x - K'_{\dot{p}}) + K'_{\delta_r} N'_{\dot{v}} Y'_p + K'_{\delta_r} N'_v Y'_{\dot{p}}$$

$$+ Y'_{\delta_r} K'_{\dot{v}} N'_p - Y'_{\delta_r} K'_v(- I'_{xz} - N'_{\dot{p}}) - Y'_{\delta_r} N'_{\dot{v}} K'_p + Y'_{\delta_r} N'_v(I'_x - K'_{\dot{p}})$$

$$- N'_{\delta_r} K'_{\dot{v}} Y'_p - N'_{\delta_r} K'_v Y'_{\dot{p}} + K'_{\delta_r}(m' - Y'_{\dot{v}})N'_p + K'_{\delta_r} Y'_v(- I'_{xz} - N'_{\dot{p}}) .$$

$$C_{\psi} = - N'_{\delta_r}(m' - Y'_{\dot{v}})K'_{\phi} + N'_{\delta_r}Y'_vK'_p + K'_{\delta_r}N'_{\dot{v}}Y'_{\phi} + K'_{\delta_r}N'_vY'_p + Y'_{\delta_r}K'_{\dot{v}}N'_{\phi}$$

$$+ Y'_{\delta_r}K'_vN'_p - Y'_{\delta_r}N'_{\dot{v}}K'_{\phi} - Y'_{\delta_r}N'_vK'_p - N'_{\delta_r}K'_{\dot{v}}Y'_{\phi} - N'_{\delta_r}K'_vY'_p + K'_{\delta_r}(m' - Y'_{\dot{v}})N'_{\phi}$$

$$- K'_{\delta_r}Y'_vN'_p \; .$$

$$D_{\psi} = N'_{\delta_r}Y'_vK'_{\phi} + K'_{\delta_r}N'_vY'_{\phi} + Y'_{\delta_r}K'_vN'_{\phi} - Y'_{\delta_r}N'_vK'_{\phi} - N'_{\delta_r}K'_vY'_{\phi} - K'_{\delta_r}Y'_vN'_{\phi} \; .$$

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